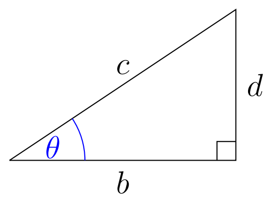
Trigonometric Substitution

Trigonometric Substitutions

Expression	Substitution	Domain	Identity
$\sqrt{a^2-x^2}$	$x=a\sin\theta$	$-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$	$1-\sin^2\theta=\cos^2\theta$
$\sqrt{a^2+x^2}$	$x=a\tan\theta$	$-\frac{\pi}{2}<\theta<\frac{\pi}{2}$	$1+\tan^2\theta=\sec^2\theta$
$\sqrt{x^2-a^2}$	$x=a\sec\theta$	$0\le\theta<\frac{\pi}{2}$ or $\pi\le\theta<\frac{3\pi}{2}$	$\sec^2\theta-1=\tan^2\theta$

Trigonometric Substitution Diagram

When solving a problem with trigonometric substitution, we may need to switch back to having things in terms of x. A triangle like the one below can help us.



Using the equation from our substitution, we can fill in our triangle.

Example 1

$\int \frac{\sqrt{9-x^2}}{x^2}dx$

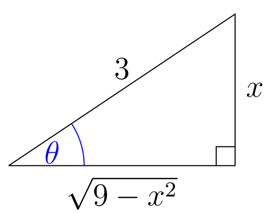
This is of the form $\sqrt{a^2-x^2}$, so we let $x=3\sin\theta$. Then $dx=3\cos\theta d\theta$.

$\int \frac{\sqrt{9-x^2}}{x^2}dx = \int \frac{\sqrt{9-(3\sin\theta)^2}}{(3\sin\theta)^2}(3\cos\theta)d\theta = \int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta}(3\cos\theta)d\theta = \int \frac{3\sqrt{1-\sin^2\theta}}{9\sin^2\theta}(3\cos\theta)d\theta$

$= \int \frac{\sqrt{1-\sin^2\theta}}{\sin^2\theta}(\cos\theta)d\theta = \int \frac{\cos\theta}{\sin^2\theta}(\cos\theta)d\theta = \int \frac{\cos^2\theta}{\sin^2\theta}d\theta = \int \cot^2\theta d\theta = \int(\csc^2\theta-1)d\theta$

$= -\cot\theta-\theta+C$

Next we need to have everything in terms of x. $x=3\sin\theta$ tells us $\sin\theta=\frac{x}{3}$. Also, $\theta=\arcsin\frac{x}{3}$. Now we can use a triangle to solve.



By our triangle, we know $\cot\theta=\frac{\sqrt{9-x^2}}{x}$.

Finally we can use substitution.

$-\cot\theta-\theta+C = -\frac{\sqrt{9-x^2}}{x}-\arcsin\frac{x}{3}+C$

Example 2

$\int \frac{1}{x^2\sqrt{x^2+4}}dx$

This fits the form of $\sqrt{a^2+x^2}$. We let $x = 2\tan\theta$. Then $dx = 2\sec^2\theta d\theta$. Next we substitute.

$\int \frac{1}{(2\tan\theta)^2\sqrt{(2\tan\theta)^2+4}}2\sec^2\theta d\theta = \int \frac{1}{(4\tan^2\theta)2\sqrt{\tan^2\theta+1}}2\sec^2\theta d\theta = \int \frac{1}{(4\tan^2\theta)2\sec\theta}2\sec^2\theta d\theta$

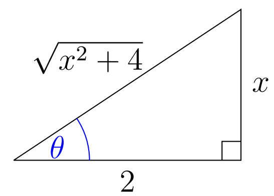
$\frac{1}{4}\int \frac{1}{(\tan^2\theta)}\sec\theta d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}d\theta$

Next we need to use u-substitution. Let $u = \sin\theta$, then $du = \cos\theta d\theta$.

$\frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}d\theta = \frac{1}{4}\int \frac{1}{u^2}du = -\frac{1}{4}\frac{1}{u} + C$

And by our u-substitution we have $-\frac{1}{4}\frac{1}{\sin\theta} + C$, or $-\frac{1}{4}\csc\theta + C$.

Next we use a triangle to put our answer in terms of x. By above, $\tan\theta = \frac{x}{2}$.



By our triangle, $-\frac{1}{4}\csc\theta + C = -\frac{\sqrt{x^2+4}}{4x} + C$.

Example 3

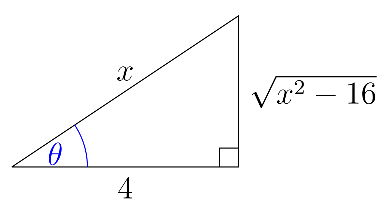
$\int \frac{\sqrt{x^2-16}}{x}dx$

This problem fits the last form, $\sqrt{x^2-a^2}$. We let $x = 4\sec\theta$, then $dx = 4\sec\theta\tan\theta d\theta$.

$\int \frac{\sqrt{(4\sec\theta)^2-16}}{4\sec\theta}4\sec\theta\tan\theta d\theta = \int \sqrt{(4\sec\theta)^2-16}\cdot(\tan\theta)d\theta = \int 4\tan\theta(\tan\theta)d\theta = \int 4\tan^2\theta d\theta$

$= 4\int(\sec^2\theta - 1)d\theta = 4\tan\theta - 4\theta + C$

By above, we have $\sec\theta = \frac{x}{4}$, $\theta = \sec^{-1}(\frac{x}{4})$, and the following:



Then our answer becomes $4\frac{\sqrt{x^2-16}}{4} - 4\sec^{-1}\left(\frac{x}{4}\right) + C = \sqrt{x^2-16} - 4\sec^{-1}\left(\frac{x}{4}\right) + C$